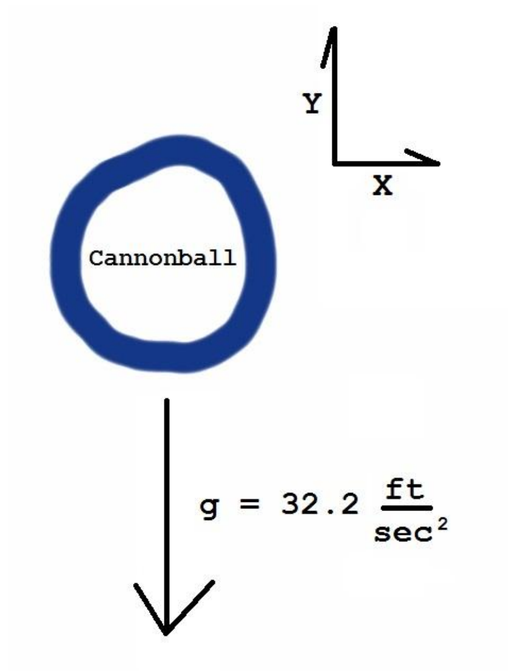
Ballistics

A free body diagram and a little calculus gives us all the equations we need to predict displacement, velocity, and acceleration of a projectile.

Ballistics is the study of projectiles in motion. It is a vast field of science that is used to design firearms, build rockets, and solve crimes. Let's derive the equations from a simple physics problem by using calculus and the relationship of displacement, velocity, and acceleration.

Displacement is the integral of velocity which is the integral of acceleration. All these are integrated with respect to time.

Let's start with a cannonball shot from a cannon. A free-body diagram is a tool used in physics to determine the resultant forces on a body. We start with a representation of the body and all the forces.



When we draw our cannonball in the air we see that there is nothing acting on the cannonball but gravity which is accelerating the ball down at a rate of the gravitational acceleration (g = 9.8 m/s^2 or 32.2 ft/s^2). There are other forces (like drag) but we are going to just deal with the gravitational force for our formula derivation. We will use the standard convention of up being the positive Y-axis (our vertical axis) and right being the positive X-axis (our horizontal axis). No force in the X-axis effects the forces in the Y-axis so we can keep these two values separate. If a force acts in both directions, we will break it into its X and Y components. Let's start by writing down the equations in the X and Y directions for our acceleration. We will show the X and Y components in parentheses with a subscript to show the corresponding axis.

Acceleration: $\quad (0)_x + (-g)_y$

This equation shows that for any time, the cannonball is accelerating down at a rate of g. Let's take the integral of acceleration to come up with velocity:

Velocity: $(C_x)_x + (-gt + C_y)_y$

When we do this, we see that it has introduced some unknown constants. The C_x represents some velocity in the x direction and the C_y represents some velocity in the y direction. We can get rid of integration constants by applying some initial conditions. Let's say at the start of the ballistics problem, at time t = 0, that the velocity in the x direction is V_{0x} and the velocity in the y direction is V_{0y}. In the above equation, if we set t=0, we see that C_x represents the velocity in the x direction which, for t=0, is V_{0x}. We can make the same argument for the y direction. Substituting in these values, the velocity equation becomes:

Velocity: $(V_{0x})_x \quad (-gt + V_{0y})_y$

This step may seem arbitrary because we replaced one constant for another. But we replaced some unknown constant for a defined value. There is a good chance that we may know the initial velocity values of a ballistics problem so we will be able to apply those in this equation.

This velocity equation is already telling us quite a bit. If you look at the x value you will notice there is no time dependent effect. Whatever velocity in the x direction we start with, that value will be the same value through the whole ballistics problem. In the y direction, we will start with the initial velocity, but more negative velocity will be added as time goes along. From this equation, we can already start to describe our ballistics. For instance, if we throw a ball up into the air, how long will it take to reach the peak? Well, we have observed that when you throw something up, it goes up, slows, and then comes down. That means there was a time when the y velocity was zero. This time is at the very peak. So, if we set the y velocity term to zero and solve for t, we will have the formula for the time to peak.

Time to Peak: $-gt + V_{0y} = 0$

$$gt = V_{0y}$$

$$\mathbf{t = V_{0y} / g}$$

We see that if we toss a ball up, the time it takes to reach its peak is its initial y velocity divided by g. Let's say a baseball pitcher can throw a 90-mph fastball and can throw it straight up. How long would it take to reach its peak?

Convert to ft/sec: (90 miles/hr)(5280 ft/mile)(1 hr/3600 sec) = 132 ft/sec

Time to peak: (132 ft/sec) / 32.2 ft/sec^2 = **4.1 secs**

That's pretty impressive. Let's integrate our velocity equation to get displacement. Displacement is the position of the projectile with respect to a reference position.

Displacement: $(V_{0x}t + C_x)_x$ $(-gt^2/2 + V_{0y}t + C_y)_y$

Again, we have some integration constants. Let's replace them with some initial conditions. Let's say at t=0 that the projectile starts at initial displacement values S_{0x} and S_{0y}. So, the final displacement equation is:

Displacement: $(V_{0x}t + S_{0x})_x$ $(-gt^2/2 + V_{0y}t + S_{0y})_y$

Often in ballistics problems, the initial displacements are the starting point of the problem and we can say at t=0, S_{0x} and S_{0y} =0.

Let's use this equation to solve for how high the baseball in the previous problem would go. We already calculated the time it would take the baseball to reach its peak. Let's use the y direction displacement formula to find the distance above ground the ball will stop.

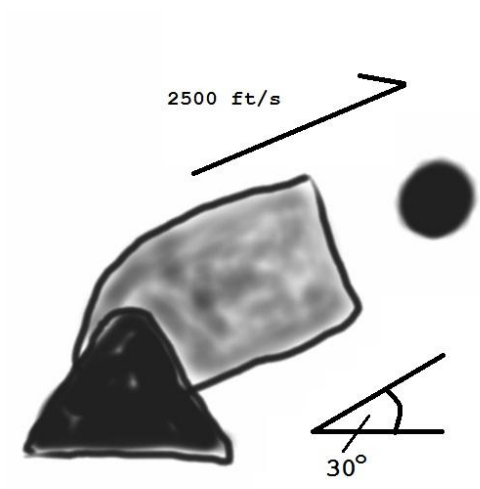
Height of ball at time t = $-gt^2/2 + V_{0y}t + S_{0y}$

We know the initial velocity (132 ft/sec) and the time to peak (4.1 sec) and can assume the ground is the initial displacement and consider it as zero.

Height of ball at peak = $-(32.2 \text{ ft/sec}^2)(4.1 \text{ sec})^2/2 + (132 \text{ ft/sec})(4.1 \text{ sec}) = 271$ **feet**

That is almost as high as a football field is long.

Let's look at our cannonball problem. Let's say we shoot a cannonball with a muzzle velocity of 250 ft/sec and aim the cannon at a 30 degree angle to the ground. Let's calculate how far the cannonball will travel on a flat plane until it hits the ground. We will be looking for the displacement in the x direction. We can quickly calculate the x and y initial velocities.



V_{0x} = (250 ft/sec) cos(30 deg) = 217 ft/sec

V_{0y} = (250 ft/sec) sin(30 deg) = 125 ft/sec

Our x direction displacement equation is:

$S_x = V_{0x}t + S_{0x}$

We can take the origin of our cannon as (0, 0) so the initial displacements will be zero. You can see from the equation that we will need to determine the time the cannonball is in flight. What do we know about the final condition of the cannonball? We know that it will rise to some value and return to a vertical displacement of zero. If we set the y direction displacement to zero, we can calculate the time it takes for the ball to hit the ground.

y displacement = $-gt^2/2 + V_{0y}t + S_{0y}$

Again, we can consider the initial displacement to be zero. When we set the above to equal zero (the final vertical displacement), we get:

$0 = -gt^2/2 + V_{0y}t$

$0 = -gt/2 + V_{0y}$

$gt/2 = V_{0y}$

$t = 2V_{0y}/g = 2(125 \text{ ft/s}) / (32.2 \text{ ft/sec}^2) =$ **7.76 sec**

So, the time from when the cannonball leaves the muzzle until it hits the ground is 7.76 secs. We can now plug that into the x-displacement equation to figure out how far that is.

$S_x = V_{0x}t = (217 \text{ ft/sec})\ 7.76 \text{ sec} =$ **1685 ft**

So, our cannonball will hit the ground 1685 feet away.

This is a typical ballistics problem. Often, you need to find a quantity, and you must use multiple equations and initial conditions to find all the data you need to solve it. But it is amazing to think that we started with just the fact that the cannonball is only affected by gravity in flight and we could use calculus and physics to derive the equations that describe the acceleration, velocity, and displacement for the projectile at any time.

