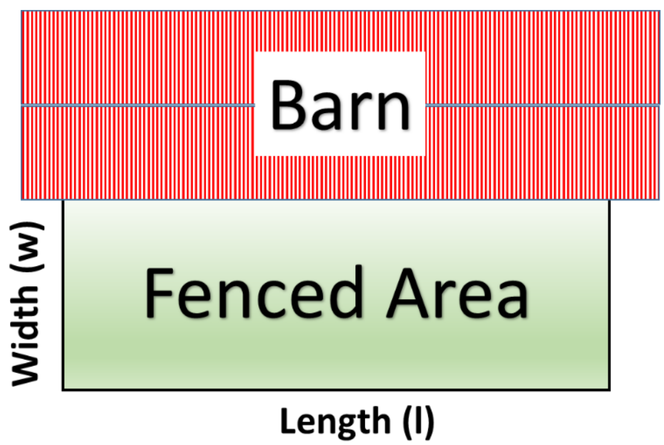
Optimization

If it can save me money, I am interested.

Did you ever sit in school and think “What is this good for?”

Well, math has many uses. But saving money is probably one of the most loved. Math gives us the ability to optimize problems. If we can develop an equation for something, then we can find maximum and minimum values for that equation as we vary the independent variable. Engineers and Scientist use these tools all of the time to find the best, shortest, or least expensive options. Let’s use an example to see how this works.

Let’s say you own a ranch and want to enclose part of your land with a fence to keep your horses. Fence is expensive, so you want to be able to minimize the amount of fence you use to minimize the cost. Ah, this already sounds like an optimization problem.



As seen in the picture above, you want to use your barn as one of the sides to minimize the amount of fence you need. And a local ordinance says you have to have at least an area of 2000 ft^2 for the horses. What you need to know is what dimensions (length and width) of the fence will allow you to meet the area requirement but minimize the amount of fence you need.

The first step is to develop an equation for the parameter you want to optimize (in this case, you want to minimize fence size). Looking at the above plan, the equation is simply:

$$\boldsymbol{Amount\ of\ fence = width + length + width = 2w + l}$$

Now, in order to optimize this equation, you have to pick a single variable to act as the independent variable. We could pick a value for width and watch how the length changes but that wouldn’t tell us a lot. Because we want to minimize the fence, if we picked a value for

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width, we would want the length to be zero to minimize the total length of fence. But that won't work for the horses. Luckily, we also have another constraint. We have to make sure the area is where we need it to be. The area of the fenced in section will be simply:

$$\boldsymbol{Area = w\ x\ l = 2000\ ft^2}$$

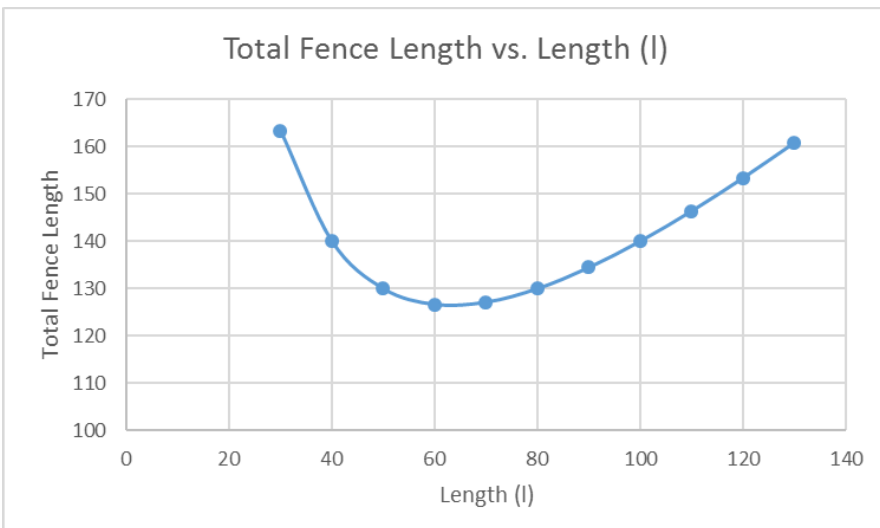
By combining these two equations we can have one equation that is in only one variable that is ready to be minimized. Let's see how that works. Let's solve the area equation for width:

$$\boldsymbol{w = \frac{2000\ ft^2}{l}}$$

Let's substitute this equation for w into the first equation:

$$\boldsymbol{Amount\ of\ fence = 2w + l = 2\left(\frac{2000\ ft^2}{l}\right) + l = \frac{4000\ ft^2 +\ l^2}{l}}$$

We now have an equation in one variable that represents the total length of fence and is constrained by our area requirement. All we have to do is find out where this equation is the smallest, and we will have arrived at the smallest amount of fence. Let's go to the computer and graph this function:



Using a spreadsheet, we can plot this function. You will see from this plot that the function definitely has a minimum value. Picking it off the chart, it looks close to when the length is 60 feet. If the length of the long side is 60 feet, let's calculate the width value from the above area formula:

$$\boldsymbol{w = \frac{2000\ ft^2}{l}} = \frac{2000\ ft^2}{60\ ft} = \boldsymbol{33.3\ ft}$$

You now have the dimensions of your fence to minimize the cost and ensure you have a minimum of 2000 ft^2 area for your horses. The area will be 60 ft by 33.3 ft and the total fence length will be 126.7 ft.

As an engineer, I am not happy with pulling a number off of the graph, so one of the tools that engineers use is calculus. Calculus allows us to find the exact minimum of a function. The derivative of a function gives its rate of change for any given point. At the minimum, the rate of change of the function is zero. If we take the derivative of the fence length function and set it equal to 0, we can solve for the value of length at the absolute minimum. Let's do it.

$$\boldsymbol{Amount\ of\ fence = \frac{4000\,ft^2 + l^2}{l}}$$

$$\boldsymbol{\frac{d}{dl} Amount\ of\ fence = \frac{d}{dl}\,\frac{4000\,ft^2 + l^2}{l}}$$

Solve and set equal to 0:

$$\boldsymbol{\frac{d}{dl} Amount\ of\ fence = \frac{-4000\,ft^2}{l^2} + 1 = 0}$$

Now solve for l:

$$\boldsymbol{l^2 = 4000\,ft^2}$$

So the length at the minimum is:

$$\boldsymbol{l = 63.25\,ft}$$

This is just a little bigger than the length we pulled off the graph, but it is the exact minimum.

That would set the area at length = 63.25 ft and the width 31.6 ft with a total fence length of 126.5 ft.

Optimization is a huge topic in science and industry. Whether you want to minimize the route to Mars or maximize the speed of a vehicle, math allows you to optimize your understanding of optimization.